Proxy Variables and Feedback Effects in Decision Making[*]

Alexander Clyde[†]

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Abstract

When using data, an analyst often only has access to proxies of the true variables. I propose a framework that models decision makers as 'flawed statisticians' who assume potentially noisy proxy variables are perfect measurements. Due to feedback from the choices into data, a notion of equilibrium is required to close the model. I illustrate the concept with applications to policing/crime and market entry. In these examples, we see that very small imperfections in the proxy variable can lead to large distortions in beliefs. I characterize all strategies that can arise as equilibria when measurement is arbitrarily close to perfect.

Keywords: Proxy Variables, Bounded Rationality, Expectations, Misspecified Models.

JEL Classification: D91

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[†]University College London

1 Introduction

Analysis of quantitative data to inform decisions is increasingly important to organizations and firms. For private companies, McAfee et al. (2012) argue that improvements in the ability of managers to measure, store and collect information about their business can result in large performance gains. However, data used in economic decision making is often an imperfect measurement or proxy of the underlying variables.

Examples of proxy variables that play an important role in driving allocation of economic resources abound. GDP per capita guides entrepreneurs and traders in assessing the relative economic vitality of countries in which they are considering investment, and is used by governments in determining important policy decisions. Yet as a proxy for living standards it has come in for criticism[1]. The use of citation metrics is another case. Governments and academic institutions find these metrics valuable for assessing academic impact, but there is debate around the extent to which they are truly good measures[2].

In this paper I propose a framework for modelling decision makers who naively use possibly mismeasured proxies. The decision makers (henceforth DMs) in this framework are assumed to form expectations about the impact of their actions from the proxy variables they have available, treating the proxy variables as if they were *exactly identical* to the true variables. This follows a tradition in economics and psychology of modelling economic agents as 'flawed statisticians', for example early work in behavioural economics on the 'law of small numbers' by Tversky and Kahneman (1971) to more recent work testing the 'What You See Is All There Is' heuristic in experiments (Enke, 2020).

The structure of the agent's problem is as follows. First they draw the realization of two signal variables (s, z). These signals consist of a circumstance variable s that affects utility but is not used in forming beliefs, and a control variable z

[1]See Coyle (2015) for an outline of various arguments in this debate.

[2]See Borchardt and Hartings (2018) for discussion and reference to work (Borchardt et al., 2018) providing evidence that for academic chemists there are significant differences between the level of citations a paper receives and perceived academic impact among scientists.

that does not affect utility directly but does affect beliefs. They then choose an action variable x, and both the action and control then affect the realization of an outcome variable y. Agents are assumed to have a vNM utility function over the outcome, action, and circumstance variables. Finally, a vector of proxy variables $(y^\bullet, x^\bullet, z^\bullet)$ is drawn from a distribution π that depends on (y, x, z), where each of the true variables has a corresponding proxy. The DMs only have access to data that gives them knowledge of the joint distribution of the proxies. Due to the possible imperfections in the variables available, choices made by these decision makers can affect the data used for belief formation. I define a notion of Proxy Equilibrium which ensures consistency between these choices and the data.

A preview of the first example in the paper can be used to illustrate this structure. Suppose the decision maker is a municipal policymaker who must decide on the number of police officers to employ. Before deciding, they first learn the realization of a circumstance variable s that affects the cost of crime. The police number x they choose then affects crime level y in the municipality. The policymaker needs to learn the relationship between their police numbers choice and crime in order to make a decision. However the policymaker can only learn the relationship from potentially noisy proxy variables $x^\bullet, y^\bullet$.

In the first example neglect of measurement error results in overly rigid policy. Due to attenuation bias in the measured relationship between policing and crime, municipal policymakers vary police numbers with the cost of crime less than they would if they knew the true relationship. This rigidity can be extreme, there exists a Proxy Equilibrium in which municipalities do not vary police numbers at all. This is true regardless of how strong the true relationship between crime and policing is. The example demonstrates the difference between a purely statistical notion of imperfection in measurement and one informed by the possibility of decision maker's choices feeding back into the data. Due to feedback effects, there is a stark discontinuity between the extent of imperfection in proxy variables and the extent of the bias in the beliefs of the DMs.

In the second example, noisy measurement and equilibrium selection effects result in endogenous overoptimism and thus over-entry by firms deciding whether to enter a market or not. Moreover, the impact of changes in measurement noise can differ in important ways for pivotal firms who are on the margin between different choices and non-pivotal firms who are not. Due to the impact on the pivotal firm who must be indifferent between entering or not, more proxy 'noise' results in a greater extent of excessive entry. Without this equilibrium feedback effect, the effect of noise on entry is ambiguous. This is in contrast to other work on the behavioural bias caused by selection effects, such as Jehiel (2018) and Esponda and Pouzo (2017), in which more noise has an ambiguous effect on entry both in and out of equilibrium.

Finally, I build on the insights in these examples and give a characterization of all strategies that could arise as Proxy Equilibria when the proxy variables are arbitrarily close to being perfect measurements. I show that a strategy can be implemented as a Proxy Equilibrium for some proxy mapping that is arbitrarily close to perfect measurement if and only if it satisfies a Self-Confirming Optimal property. Roughly, a strategy is Self-Confirming Optimal if it is optimal against beliefs that coincide with rational expectations for at least any action-control signal combinations that occur with positive probability under the strategy. The characterization result clarifies a theme arising in the two examples; that small measurement problems can have a large effect on beliefs when the equilibrium strategy only puts weight on particular actions.

The concept draws a distinction between the fact that the DM knows the realization of the signals and the action they have chosen but does not know how these variables covary with the outcome they wish to forecast. The story I have in mind is that the joint distribution over proxies is generated as a long run steady state of some learning process. The learning process is not that of a long lived agent who repeats the same decision problem enough times to generate an asymptotic sample, but instead a short-lived agent who does not generate enough experiences of the effect of their own action and signals and has to rely on a large public dataset of potentially mismeasured proxies. For concreteness, in the policing example we can imagine a sequence of short lived municipal leaders. The data generated from each municipal leader's tenure is too sparse to apply the law of large numbers, so the leaders have to draw inference from the experiences of other municipalities in other time periods by using a national dataset designed for social scientists researching crime.

The contribution of this paper is twofold. First, it contributes to the literature on solution concepts with bounded rational expectations by considering issues of measurement and proxies in an equilibrium framework. The concept generally does not fall neatly in others in the literature, and I explore these connections in Section 6. The concept demonstrates how issues of belief distortion similar to that explored in the growing literature on misspecified models can occur purely as a result of the neglect of measurement noise. Secondly, it develops applications of the concept to a variety of settings in which organizations use data to form beliefs. It shows how issues arising from calculated decision making feeding back into data can be important considerations for evaluating the use of statistical information by economic organizations.

2 Modelling Set Up

The space of variables V can be divided into four dimensions. There is a set of circumstance signals $S \subseteq \mathbb{R}$ with realization s, a set of control signals $Z \subseteq \mathbb{R}$ with realization z, a set of actions $X \subseteq \mathbb{R}$ with realization x and a set of outcomes $Y \subseteq \mathbb{R}$ with realization y. The variable space can thus be described as $V = Y \times X \times Z \times S$. The circumstance signal represents private payoff shocks, while the control signal represents shocks for which there is data on how they vary with actions and outcomes.

The idea will be that our decision maker learns the realization of the circumstance variable s and the control variable z, before choosing an action x resulting in a distribution over the outcome variable y. The payoff of the decision maker (henceforth DM) is defined over the circumstance, action and outcome variables by utility function $u : Y \times X \times S \to \mathbb{R}$. In examples, when a variable only takes on a single value we suppress that variable in notation.

Throughout this paper, given a variable space V when we say a distribution admits a density we assume that this is with respect to the same σ-finite product measure. In all examples in this paper this will either be the counting measure for the finite case or Lebesgue measure for the continuum case. The causal structure between the variables is represented by the graph in Figure 2. The control and circumstance variables affect the action variable, and in turn the control and action variables affect the outcome variable.

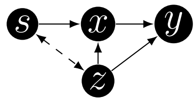

Figure 1: Causal Structure

The DM wants to form the conditional distribution $p(y|x,z)$. Doing this, they can calculate their objective expected utility:

$$U(x,z,s) \equiv \int_Y u(y,x,s)p(y|x,z)d\mu(y) \tag{1}$$

Throughout the paper we treat choosing actions to maximize this expression as the normative benchmark. Let $\sigma : Z \times S \to \Delta(X)$ denote a *strategy mapping*. Assume the exogenous joint distribution over controls and circumstances is full support and admits a density $p(z,s)$. Then given any Borel Set $\mathcal{X} \in B(X)$, we can write a strategy mapping conditional only on z.

$$\sigma(\mathcal{X}|z) = \int_S \sigma(\mathcal{X}|z,s)p(s)d\mu(s) \tag{2}$$

Denote any Borel subset of the variable space by $\mathcal{W} \equiv \mathcal{Y} \times \mathcal{X} \times \mathcal{Z} \in B(Y) \times B(X) \times B(Z)$[3]. Let $P(\cdot;\sigma) \in \Delta(V)$ be the objective distribution over the variables. Using the causal structure, the joint distribution over outcomes, actions

[3]Since the variable spaces are subsets of $\mathbb{R}$ and therefore second countable, the Borel product σ-algebra is equal to the product of the Borel σ-algebra for each dimension.

and controls can be written as follows.

$$P(\mathcal{Y}, \mathcal{X}, \mathcal{Z}; \sigma) = \int_{\mathcal{Y} \times \mathcal{Z}} [\int_{\mathcal{X}} p(y|x, z) d\sigma(x|z)] p(z) d\mu(z, y) \tag{3}$$

2.1 Proxy variables

In order to form beliefs about the distribution of outcomes conditional on a given action being taken when a given control is realized, the DM needs to have information on the joint distribution of (y, x, z). We assume that the DM can only access the joint distribution over proxies for these variables. Each of the three variables has a respective proxy that can take any of the values the variable it is a proxy for can take. We denote a realization of the proxy for the outcome, action and control by $(y^\bullet, x^\bullet, z^\bullet) \in Y \times X \times Z$ respectively. We define a *proxy mapping* $\pi : Y \times X \times Z \to \Delta(Y \times X \times Z)$ that induces a distribution over the proxies for any realization of the true variables. For any Borel subset $\mathcal{W} \in B(Y) \times B(X) \times B(Z)$, the induced distribution over proxy variables is then:

$$P_\pi(\mathcal{W}; \sigma) = \int_{Y \times X \times Z} \pi(\mathcal{W}|y, x, z) dP(y, x, z; \sigma) \tag{4}$$

Assume that the proxy mapping and strategy are such that the induced distribution over proxies P_π admits a density p_π. In order to form beliefs about how their action affects the distribution over outcomes, the DM needs to form conditional beliefs:

$$p_\pi(y^\bullet|x^\bullet, z^\bullet; \sigma) = \frac{p_\pi(y^\bullet, x^\bullet, z^\bullet, ; \sigma)}{p_\pi(x^\bullet, z^\bullet; \sigma)} \tag{5}$$

Given the distorted belief distribution, the agent in circumstance s with control z chooses an action x to maximize the perceived utility given below.

$$V(x, z, s; \sigma) = \int_{Y^\bullet} u(y = y^\bullet, x, s) p_\pi(y^\bullet|x^\bullet = x, z^\bullet = z; \sigma) \mu(y^\bullet) \tag{6}$$

2.1.1 The proxy mapping

Consider that i is any of the three variables, and that we can denote a realization of the true variable by v_i and a realization of the proxy by $v_i^\bullet$. It is possible that a proxy is a perfect measure for the underlying true variable, $v_i^\bullet = v_i$ almost everywhere for some variable(s) i. Indeed, for all the applications in this paper some of the variables are perfectly observed. In the case where $v_i^\bullet \neq v_i$ with nonzero probability, we say that $i^\bullet$ is a *mismeasurement* of i. In examples, for simplicity we generally avoid drawing a distinction between the true and proxy variable when the proxy is a perfect measurement.

A proxy mapping that induces an identical joint distribution over the proxy variables and the true variables —for any initial distribution of the true variables— is called the *perfect measurement* mapping. Throughout the paper, we say that the beliefs induced by the perfect measurement mapping comprise the *rational expectations benchmark* or induce *correct beliefs*. In the absence of a knowledge of the true relationship between variables a rational bayesian DM would have to form a prior belief about how the proxies and the true variables relate. From a normative perspective it is unclear how such a belief should be formed.

In our examples, we illustrate the causal dependencies between the variables by Directed Acyclic Graphs (DAGs). In these graphs, a link $\rightarrow$ between two variables indicates that the variable being pointed to is independent of all other variables conditional on the variables that point into it. DAGs are used to model causal misperceptions in the related concept of Spiegler (2016), and we discuss the relationship between the two concepts further in Section 6.2.

2.2 Equilibrium

We see how the decision maker's strategy can affect expectations by considering the case where all variables are finite. For any proxy mapping π we can write the perceived conditional distribution as follows.

$$p_\pi(y^\bullet|x^\bullet, z^\bullet; \sigma) = \frac{\sum_{y,x,z} \pi(y^\bullet, x^\bullet, z^\bullet|y, x, z)p(y|x, z)\sigma(x|z)p(z)}{\sum_{y^\bullet,y,x,z} \pi(y^\bullet, x^\bullet, z^\bullet|y, x, z)p(y|x, z)\sigma(x|z)p(z)} \tag{7}$$

We illustrate the dependence of this distribution on the strategy $\sigma(x|z)$ using the following binary version of the policing example previewed in the introduction.

Example 1. *A municipal leader learns the realization of a circumstance variable s determining whether the cost of crime is high $s = \bar{s}$ or low $s = \underline{s} < \bar{s}$, before choosing whether to hire more police officers $x = 1$ or not $x = 0$. The hiring of police officers in turn affects whether crime is high $y = 1$ or low $y = 0$.*

Let the relationship between the policing variables and the crime variable be given by $p(y = 1|x) = \beta x + (1 - \beta)(1 - x)$ where $\beta \in (0, \frac{1}{2})$. The prior distribution over the cost of crime variable is $p(\bar{s}) = \frac{1}{2}$. We assume the crime variable y is perfectly measured, but the policing variable x is potentially not. We can write the simplest form of measurement error in the policing variable using the proxy mapping $\pi_x(x^\bullet = x|x) = \lambda$ where $\lambda \in (\frac{1}{2}, 1]$. As $\lambda \to 1$ we have close to perfect measurement.

Denote the ex-ante strategy as $\sigma(1) = \frac{1}{2}\sigma(1|\bar{s}) + \frac{1}{2}\sigma(1|\underline{s})$. The perceived conditional distribution is then:

$$p_\pi(y^\bullet = 1|x^\bullet = 1; \sigma) = \frac{\sigma(1)\lambda\beta + \sigma(0)(1 - \lambda)(1 - \beta)}{\sigma(1)\lambda + \sigma(0)(1 - \lambda)} \tag{8}$$

This expression is clearly not invariant to the strategy. For example if $\sigma(1) = \sigma(0)$ it is equal to $\lambda\beta + (1 - \lambda)(1 - \beta)$ while if $\sigma(1) = 1$ and $\sigma(0) = 0$ it is equal to β.

Thus, in general to characterize the DM's choices we need to define an equilibrium concept in order to establish consistency between strategies and beliefs. To ensure that conditional distributions are well defined we first define an equilibrium with a small trembling probability and then define an equilibrium as the limit when this probability goes to zero.

We make the following technical definitions to facilitate the description of Proxy Equilibrium. We say a sequence of strategies $\{\sigma\}_{j=1}^\infty$ *converges* to strategy $\bar{\sigma}$ if for every $(z, s) \in Z \times S$ the sequence of probability measures $\{\sigma(.|z, s)\}_{j=1}^\infty$ converges in distribution to the probability measure $\bar{\sigma}(.|z, s)$. A strategy σ *in-*

duces full support if it induces a belief density such that $p_\pi(x^\bullet, z^\bullet; \sigma) > 0$ for any realization $(x^\bullet, z^\bullet) \in X \times Z$.

Definition 1. *Let σ^*_ϵ be a strategy mapping that induces full support. For every $s \in S$, $z \in Z$, define the following set:*

$$X(z, s; \sigma^*_\epsilon) \equiv$$
$$\{x \in X : x \notin \arg\max \int_{Y^\bullet} u(y = y^\bullet, x, s) p_\pi(y^\bullet | x^\bullet = x, z^\bullet = z; \sigma^*_\epsilon) d\mu(y^\bullet)\}$$

*Then σ^*_ϵ is an ϵ-**Proxy Equilibrium** if for every Borel subset $I \in B(X)$, $I \subseteq X(z, s; \sigma^*_\epsilon)$, we have that $\sigma^*_\epsilon(I|z, s) < \epsilon$.*

Definition 2. *A strategy σ^* is an **Proxy Equilibrium** if there exists a sequence $\{\sigma^*_l\}_{l=1}^\infty$ converging to σ^* as well as a sequence $\epsilon^l \to 0$, such that for every l, σ^*_l is an ϵ^l-Proxy Equilibrium.*

The definition ensures that action-control-circumstance combinations that are not in the best response correspondence must have vanishing probability under the equilibrium strategy. Note that it is a requirement of the definition of an ϵ-Proxy Equilibrium for the equilibrium strategy to induce a distribution over proxies that admits a density.

When the variable space is finite and the proxy mapping satisfies a *minimal responsiveness* condition, we can show the existence of at least one Proxy Equilibrium using conventional methods. A proxy mapping π is minimally responsive if whenever the conditional-on-z strategy mapping admits a density and is full support: *supp*$\{\sigma(.|z)\} = X$ for all $z \in Z$; then we have $p_\pi(x^\bullet, z^\bullet; \sigma) > 0$ for any realization $(x^\bullet, z^\bullet)$.

Proposition 1. *Assume the set V is finite and the proxy mapping π is minimally responsive. Then a Proxy Equilibrium exists.*

Proof. In Appendix □

2.3 Imperfect observations of actions

The formalism allows the DM to have imperfect equilibrium knowledge of their own distribution of actions. This is motivated by a population level interpretation, where the dataset the DM is using to form beliefs is generated by other DMs facing the same or similar decision problems. The DM's own record of past actions only makes up a negligible part of this dataset. For example, a DM in a particular city may draw inference from a national level dataset when inferring the effect of a policy action on an outcome.

This allowance differs from the requirement of knowledge of the distribution of own actions in other solution concepts in the literature, such as Berk-Nash Equilibrium (Esponda and Pouzo, 2016) or Self-Confirming Equilibrium (Fudenberg and Levine, 1993). We discuss the substantive difference this can make in Section 6.

3 An illustrative example: Rigid Policing

In this example we consider the leader of a municipal authority, who has to make a decision on the number of police officers to hire. The municipal leader wants to hire police officers in order to reduce crime. We assume that there is noise in the measured variable for police numbers. Concern about measurement error in police staffing figures is not unprecedented. It is argued by Chalfin and McCrary (2018) that based on discrepancies between official data and administrative and census information there is significant measurement error in police staffing numbers in the literature estimating the effect of police numbers on crime. For expositional purposes, we assume that the crime variable is measured perfectly. This turns out not to make a difference to the equilibria that we characterize.

The structure of the problem facing the municipal leader is as follows, first they learn the realization of a variable affecting the cost of crime in their municipality s. This is assumed to be distributed normally in the population of municipalities used in the dataset under consideration, $s \sim \mathcal{N}(0, \sigma_s^2)$. The municipal leader then chooses the change in the number of police officers. This affects the change in crime observed under their leadership via the relationship $y = \alpha - \beta x + u$, where $u \sim \mathcal{N}(0, \sigma_u^2)$ and $\beta > 0$. In the available data, it is assumed that changes in police numbers are measured as $x^\bullet = x + \epsilon$, where ϵ is normally distributed measurement error $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$. The relationship between the variables can be characterized by the DAG in Figure 2.

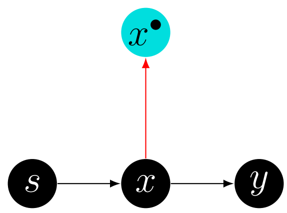

Figure 2: Crime and Policing

The utility function of the municipal leader trades-off crime and policing costs. Higher s is assumed to reflect higher costs of crime relative to altering police numbers. We assume a symmetric cost of hiring and firing police officers. An alternative interpretation is that all the variables are in logs of the levels.

$$u(y, x, s) = -s \cdot y - \frac{1}{2}x^2 \tag{9}$$

Denote the rational expectations benchmark for how policing affects crime levels in expectation by $\mathbb{E}[y|x] = f(x) = \alpha - \beta x$. We can see that by plugging this into the utility function and calculating the best response that the optimal strategy under rational expectations for the municipal leader is to set police numbers such that $x^*(s) = \beta s$. Thus the rational expectations benchmark is for police numbers to be increased by more when the costs of crime is larger (higher s) and the effect of police numbers on crime is greater (higher $|\beta|$). Define a *linear equilibrium* as an equilibrium in which the strategy of the policy maker can be expressed as a linear function of the cost variable, $x(s) = \theta_0 + \theta_1 s$ for some $(\theta_0, \theta_1) \in \mathbb{R}^2$. We can characterize all the linear equilibria of the model as follows.

Proposition 2. *There is always a linear equilibrium in which the municipal leader never changes police numbers, with best response* $x^{nv}(s) = 0$.

In addition, if $|\beta| \geq 2\frac{\sigma_\epsilon}{\sigma_s}$*, then there exist two additional linear equilibria, with best response* $x^-(s) = (\frac{1}{2}\beta - \frac{1}{2}\sqrt{\beta^2 - 4\frac{\sigma_\epsilon^2}{\sigma_s^2}})s$ *and* $x^+(s) = (\frac{1}{2}\beta + \frac{1}{2}\sqrt{\beta^2 - 4\frac{\sigma_\epsilon^2}{\sigma_s^2}})s$*.*

There are no other linear equilibria.

Proof. In Appendix □

Due to the measurement error in the police numbers proxy, there is generally downward attenuation bias in the municipal leader's estimate of the expected change in the level of crime for any given change in police numbers. However, when there is more variation in police staffing numbers the measurement error is a smaller fraction of the total variance of the proxy. This means the downward attenuation effect is lessened compared to when there is little or no variation in true police staffing numbers. This effect generates potential multiplicity of equilibria.

We illustrate the solution method and equilibria in Figure 3 below. We have the following expression for the marginal effect of increasing police numbers, given the expectations induced by the strategy $x(s) = \theta_1 \cdot s$:

$$-\frac{\partial \mathbb{E}[y^\bullet | x^\bullet]}{\partial x^\bullet} = \frac{\beta \theta_1^2 \sigma_2^2}{\theta_1^2 \sigma_2^2 + \sigma_\epsilon^2} = g(\theta_1; (\beta, \sigma_s, \sigma_\epsilon)) \tag{10}$$

An equilibrium best response has to be such that $\theta_1 = g(\theta_1; (\beta, \sigma_s, \sigma_\epsilon))$. The figure below shows how this equation characterises the equilibria for two different parameter sets. We have a case in which the only equilibrium is the no variation equilibrium and a case in which all three equilibria exist.

Adding normally distributed measurement error to the crime variable, so that $y^\bullet = y + v$ with $v \sim \mathcal{N}(0, \sigma_v^2)$, does not change either the set of Proxy Equilibria nor does it change the rational expectations benchmark. That the rational expectations benchmark is unchanged is easy to see due to linearity of expectations. The proxy-equilibrium case is due to both the linearity of the conditional expectation and the fact that the additional variance in $y^\bullet$ does not affect the marginal perceived incentive over the policing variable.

One of the equilibria has the municipalities not varying police numbers at all, regardless of how great the effect of policing on crime is. This extreme, no

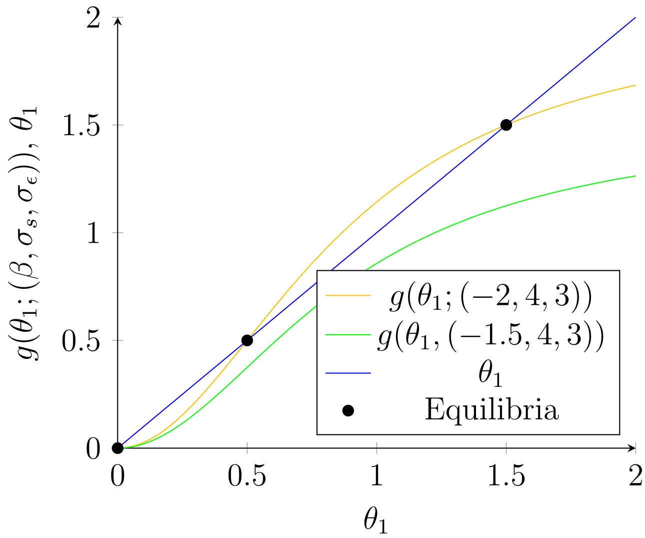

Figure 3: Best response quadratic

variation equilibrium exists for any small amount of measurement error in the proxy $\sigma_\epsilon^2 > 0$ no matter how close to zero. In this equilibrium, there is no variation in the true policing variable. Therefore even arbitrarily small noise dominates the observations of the proxies for all non-zero policing choices. This results in massive attenuation bias, so that altering police numbers appears completely ineffective at affecting crime, reinforcing that municipalities do not alter police numbers.

In all linear Proxy Equilibria, the policy chosen is more rigid than the rational benchmark. The rigidity results solely from bias beliefs due to measurement noise, and as discussed above extreme rigidity can coincide with very small noise. Apparent over-rigidity in choices has been discussed in relation to monetary policy Tetlow and Von zur Muehlen (2001) and firm pricing Nakamura and Steinsson (2013). It is possible that this example could be extended into a more complicated model that could provide an alternative explanation of these phenomena as resulting from use of mismeasured data from decision makers.

4 Market Entry: Endogenous Overoptimism

Businesses entering into new markets have high rates of failure. Using data from the US Census Bureau Haltiwanger (2015) calculates that half of new firms exit the market within 5 years. In UK data, 38 percent of enterprises newly born in 2016 survived 5 years[4]. A literature in business and economics attributes these seemingly excessive levels of market entry to overoptimism on the part of the potential market entrants, see Hayward et al. (2006), Cooper et al. (1988), Malmendier and Tate (2005).

We build an application of our solution concept that generates firms that have an upwardly biased assessment of the payoffs from entering new markets as a feature of equilibrium. Firms draw on noisily recorded data drawn from past entrants. There is a variable $z \in [0,1] \equiv Z$, representing the location of markets in some space, which could be geographical or based on demographic information. We assume that the circumstance signal only takes on a single value, and can therefore be ignored in our analysis. This means that all variation in action choice in equilibrium is driven by differences in the perceived impact of actions on outcomes. After learning the realization of this variable, the potential market entrant has to make a binary decision on whether to enter $x = 1$ or not $x = 0$. The payoff of the entrant is measured via an outcome variable $y \in \mathbb{R} \equiv Y$ representing the profitability of the enterprise, so that $u(y, x, s) = y$. The outcome variable is determined by both the entry decision and the market location variable by the following relationship.

$$\mathbb{E}[y|x,z] = \int_Y yp(y|x,z)d\mu(y) = \begin{cases} m(z) \text{ if } x = 1 \\ 0 \text{ if } x = 0 \end{cases} \tag{11}$$

We assume that the function $m : Z \to \mathbb{R}$ is strictly increasing, bounded and right-continuous, with a single point of crossing $\alpha \in [0,1]$ such that $m(z) < 0$ for all $z \in [0, \alpha)$ and $m(z) \geq 0$ for $z \in [\alpha, 1]$. Thus for high enough realizations of the market location variable, the expected profitability of entry is always greater than the payoff of zero from not entering. We assume the potential entrant does not have data on how the market location z varies with the outcome and action variable, but instead has access to a noisy recorded proxy variable $z^\bullet$. The idea

[4] This statistic is from Office for National Statistics (2022).

is that each market has a very granular definition, and in data it can only be recorded in an imprecise fashion. This could be due to data protection reasons when z is demographic information, for example.

Thus we assume the proxy variable is generated by a mapping that has the following 'window' form. There is some parameter $h \in (0, \frac{1}{2})$ such that for every $z \in [h, 1-h]$ we have that $z^{\bullet}$ is uniformly distributed on $[z-h, z+h]$. For all $z \in [0, h)$ we have that $z^{\bullet}$ is distributed uniformly on $[0, 2h)$ and for all $z \in (1-h, 1]$ we have that $z^{\bullet}$ is distributed uniformly on $z \in (1-2h, 1]$. This window form of proxy noise is similar to the notion of similarity used in Steiner and Stewart's (2008) model of learning in games. The conditional independence relationships between the true variables and the proxy are illustrated in the DAG below.

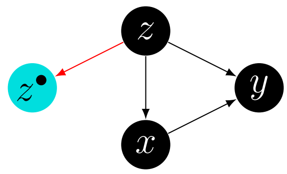

Figure 4: Market Entry

Under rational expectations the best response of the potential entrant is clear; when $z \in [0, \alpha)$ $x = 0$ is optimal while for $z \in [\alpha, 1]$ the payoff from entering is weakly above zero and therefore optimal. We are going to see that in equilibrium, there is over-entry. Since in the equilibrium data there is negligible observations of entrants below a certain market location, the proxy observations for these markets are disproportionately higher location markets that have been misclassified as lower ones. This leads to an overestimate of the payoff from entering at these lower levels. However, enough over-entry reduces the extent of this proxy bias and in equilibrium the DM is indifferent between entering or not at some cut-off $\bar{z}$ below the cut-off they would enter at under rational expectations.

Given an induced perceived distribution over the outcome variable $p_{\pi}(y|x, z^{\bullet})$, we give an expression for the perceived expected utility below.

$$\begin{aligned}
V(x=1,z;\sigma) &= \int_Y y p_\pi(y|x=1,z^\bullet=z;\sigma)d\mu(y) \\
&= \int_Y y[\int_Z p(y|x=1,\tilde{z})p_\pi(\tilde{z}|x=1,z^\bullet=z;\sigma)d\mu(\tilde{z})]d\mu(y) \\
&= \int_Z [\int_Y yp(y|x=1,\tilde{z})d\mu(y)]p_\pi(\tilde{z}|x=1,z^\bullet=z;\sigma)d\mu(\tilde{z}) \\
&= \int_Z m(\tilde{z})p_\pi(\tilde{z}|x=1,z^\bullet=z;\sigma)d\mu(\tilde{z}) \qquad (12)
\end{aligned}$$

The perceived utility of $x=0$ at z is always zero; $V(x=0,z;\sigma)=0$. We can see that the perceived utility depends on the distribution $p_\pi(\tilde{z}|x=1,z^\bullet=z;\sigma)$ induced by the strategy σ. Given the distribution over proxies, this can be calculated as follows.

$$\begin{aligned}
p_\pi(z|x=1,z^\bullet;\sigma) =& \frac{\pi(z^\bullet|z)\sigma(x=1|z)p(z)}{\int_Z \pi(z^\bullet|\hat{z})\sigma(x=1|\hat{z})p(\hat{z})d\mu(\hat{z})} \\
=& \frac{1_{[z\in[\max\{z^\bullet-h,0\},\min\{z^\bullet+h,1\}]}p(z)\sigma(x=1|z)}{\int_{\max\{z^\bullet-h,0\}}^{\min\{z^\bullet+h,1\}} p(\hat{z})\sigma(x=1|\hat{z})d\mu(\hat{z})} \qquad (13)
\end{aligned}$$

The following result shows both that Proxy Equilibria exist in this setting and that any Proxy Equilibrium with entry will take a cut-off form. We assume $h<\min\{\frac{\alpha}{2},\frac{1-\alpha}{2}\}$. This ensures h is small enough so that that the firm will not enter at low values of z and there is always an equilibrium in which the firm will enter at high values of z. The absence of this requirement on bandwidth parameter h complicates the analysis by requiring us to consider additional cases, but does not create a fundamental difference in that we can regard these cases as cut-off equilibria with cut-off points at the boundary.

Proposition 3. *Assume $h<\min\{\frac{\alpha}{2},\frac{1-\alpha}{2}\}$. Then there is a cut-off $\bar{z}\in[h,1-h]$ such that there is a Proxy Equilibrium with strategy $\sigma(x=1|z)=0$ for $z\in[0,\bar{z}]$ and $\sigma(x=1|z)=1$ for $z\in(\bar{z},1]$.*

This cut-off is always strictly less than the rational entry point; $\bar{z}<\alpha$.

In addition, there is always a Proxy Equilibrium where $\sigma(x=1|z)=0$ for all

$z \in [0, 1]$.

There are no other Proxy Equilibria.

Proof. In Appendix $\square$

In the next result we see that noisier proxies —in the form of higher h— always lead to greater levels of excessive entry.

Proposition 4. *Consider two noise parameters* $\min\{\frac{\alpha}{2}, \frac{1-\alpha}{2}\} > h_2 > h_1 > 0$. *We have that the cut-off* $\bar{z}(h_2)$ *under the positive entry Proxy Equilibrium with noise parameter* h_2 *is strictly less that the cut-off* $\bar{z}(h_1)$ *under* h_1.

Proof. In Appendix $\square$

The intuition for this result is as follows. In general —for a fixed belief distribution— it is ambiguous whether larger h increases or decreases the payoff from entry at any given z. However, in equilibrium what matters is the beliefs at the pivotal cut-off $\bar{z}$. At the cut-off the DM must be indifferent between entering and not. An increase in h will always lead to greater weight on the part of the function $m(z)$ that is above the cut-off, in particular greater weight on the positive part of $m(z)$. This pushes up the expected payoff from entering strictly above zero at this cut-off, and the new cut-off at the larger h must be below in order to restore indifference.

As a corollary to Proposition 3 and 4, we have that as $h \to 0$, the cut-off equilibria converges to the rational entry strategy. This is because $\bar{z}(h) < \alpha$ and $\bar{z}(h)$ is strictly decreasing in h.

Corollary 4.1. *Consider any sequence of bandwidth parameters* $\{h_l\}_{l=1}^{\infty}$, *such that* $0 < h_l < \min\{\frac{\alpha}{2}, \frac{1-\alpha}{2}\}$ *for all* l *and* $h_l \to 0$. *Then the corresponding sequence of Proxy Equilibrium cut-offs* $\{\bar{z}(h_l)\}_{l=1}^{\infty}$ *is such that* $\bar{z}(h_l) \to \alpha$ *from below.*

The equilibrium requirement is vital for Proposition 4. In an alternative model where the distribution $p(z)$ is exogenous, we can construct cases in which entry is both excessive and increasing h results in firms choosing to enter at fewer signals. This is because against a fixed distribution increasing h also increases the weight placed on the negative part of the $m(.)$, and this can dominate the increasing weight placed on the positive part depending on the shape of $m(.)$. We present an example where this is the case in Appendix A.1.

5 Almost Perfect Proxies

In this section we characterize the set of all strategies that can arise as Proxy Equilibria even as the variables are arbitrarily close to being perfectly measured. We call these strategies *Self-Confirming Optimal*. These strategies are optimal against any perceived beliefs that are correct 'on-path'. That is to say, correct for actions-control signal combinations that occur with positive probability under the strategy itself. If a strategy is in the set, then we can choose a particular proxy mapping that implements that strategy as an equilibrium. If a strategy does not meet the conditions to be Self-Confirming Optimal, then it cannot be implemented as a Proxy Equilibrium for any proxy mapping that is above a certain level of proximity to perfect measurement.

5.1 Definition of almost perfect proxies

We use the following concept of statistical distance to define a notion of proximity of the proxy variables to perfect measurement. The total variation distance between probability measures Q_1 and Q_2 on measure space $(\Omega, \mathcal{A})$ is:

$$TV(Q_1, Q_2) = \sup_{A \in \mathcal{A}} |Q_1(A) - Q_2(A)| \tag{14}$$

Denote $w = (y, x, z)$ and define the *perfect measurement proxy mapping* as $\pi_\delta : Y \times X \times Z \to \Delta(Y \times X \times Z)$ such that $P_\pi(W) = \int_{Y \times X \times Z} \pi_\delta(W^\bullet = W|w)p(w)d\mu(w) = P(W)$ for every Borel set $W \in B(Y) \times B(X) \times B(Z)$ and any P. We then have the following definition.

Definition 3. *Given $\eta > 0$, we say the proxy distortion mapping π is strongly*

η-close to perfect if:

$$\sup_{w \in Y \times X \times Z} TV(\pi(.|w), \pi_\delta(.|w)) < \eta \tag{15}$$

In addition, we give the following requirement for a distribution over $Y \times X \times Z$ to have full support.

Definition 4. *A distribution G over variables in $Y \times X \times Z$ is said to have **full support** if it admits a density $g(\tilde{y}, \tilde{x}, \tilde{z})$ such that $g(\tilde{x}, \tilde{z}) > 0$ for every realization $(\tilde{x}, \tilde{z}) \in X \times Z$.*

We state a result demonstrating why our notion of proximity is suitable for the Proxy Equilibrium setting. It shows that if the joint density over the true variables satisfies the full support requirement, then beliefs become arbitrarily close to rational expectations as the proxy variables become close to perfect measurements in the total variation distance. This may not hold for alternative distance requirements, such as weak convergence, which do not ensure that densities converge even as the distribution does.

The result concerns potentially out of equilibrium beliefs, and can be used as a diagnostic when considering equilibria in which full-support does not hold. For example, in our policing application the full-support assumption does not always hold and therefore we can have large belief distortions even as the proxy noise is close to zero.

To show the connection to the policing example, we give the following weaker definition of almost perfect proxies. It requires proximity to perfect measurement only for a fixed distribution over the true variables. In the appendix, we show that the stronger definition implies the weak definition holds for any distribution over the true variables.

Definition 5. *Given $\eta > 0$, we say the proxy mapping π is η-close to perfect given the distribution over true variables P if:*

$$TV(P, P_\pi) < \eta \tag{16}$$

The full support requirement rules out zero probability events in the denominator of conditional probabilities, which then ensures the convergence of the joint distribution is passed through into the conditional density. We can then show that a continuity property holds for the perceived conditional distribution of y given (x, z).

Proposition 5. *Assume the* ***full support assumption*** *holds for the true distribution P.*

Then for μ almost every $(y, x, z) \in Y \times X \times Z$, for any $\epsilon > 0$, there exists an $\eta > 0$ such that if the proxy mapping π is ***η-close to perfect*** *given true distribution P and induces a distribution over the proxy variables that satisfies the* ***full support assumption****, then $|p_\pi(y^\bullet = y|x^\bullet = x, z^\bullet = z) - p(y|x, z)| < \epsilon$.*

Proof. In Appendix □

It is clear that the policing example does not satisfy the full support requirement when the no variation equilibrium strategy is played. However, in the cases where full support holds then Proposition 5 holds and we have beliefs that are close to rational expectations for proxies that are close enough to perfect measurements. We can use the Hellinger distance, convergence in which implies convergence in the Total Variation distance, to derive an expression for the distance between any joint gaussian for proxies and true variables. For example, we can obtain an expression for the square of the Hellinger distance between the distribution for the policing variable and its proxy, which are distributed $x \sim \mathcal{N}(\mu_x, \sigma_x^2)$ and $x^\bullet \sim \mathcal{N}(\mu_x, \sigma_x^2 + \sigma_\epsilon^2)$ respectively.

$$H^2(P_\pi^{\mathbf{x}}, P^{\mathbf{x}}) = 1 - \sqrt{\frac{2\sigma_x\sqrt{\sigma_x^2 + \sigma_\epsilon^2}}{2\sigma_x^2 + \sigma_\epsilon^2}} \tag{17}$$

We can then see that as $\sigma_\epsilon \to 0$ we can make the two distributions arbitrarily close and thus satisfy our requirements for Proposition 5.

5.2 Characterization result

We define the conditions required for a strategy to be *Self-Confirming Optimal* below. The definition requires that the strategy meets different conditions for actions that are in the support of the strategy and actions that are not. Recall that $\sigma(.|z)$ is the conditional on z strategy obtained from the full strategy $\sigma(.|z,s)$ by (2). Define a *system of beliefs* as a collection of conditional distributions $q : X \times Z \to \Delta(Y)$, $q = \{q(.|x,z)\}_{(x,z)\in X\times Z}$.

Definition 6. *Let $Y \times X \times Z \times S$ be finite. A strategy $\sigma^* : Z \times S \to \Delta(X)$ is **Self-Confirming Optimal** if there exists a system of beliefs q such that:*

1. *The beliefs are correct for actions and signals that arise with positive probability under σ^*. At every (x,z) such that $x \in supp\{\sigma^*(.|z)\}$, we have $q(y|x,z) = p(y|x,z)$ for all $y \in Y$.*
2. *The strategy σ^* is optimal given beliefs q. At every $(z,s) \in Z \times S$, for any $x \in \sigma^*(.|z,s)$ and $x' \in X$ we have that:*

$$\sum_{y\in Y} u(y,x,s)q(y|x,z) \geq \sum_{y\in Y} u(y,x',s)q(y|x',z) \tag{18}$$

*If there exists q such that the first condition holds and (18) always holds strictly, we say that σ^* is **strictly Self-Confirming Optimal**.*

The strength of this condition varies with the utility function. If payoffs are such every action is a best response for some circumstance signal regardless of beliefs, then only the strategies that are optimal against the true conditional distribution are Self-Confirming Optimal. If any action can be sub-optimal at all signals for some beliefs, then the condition is very permissive. We have the following result.

Proposition 6. *Let $Y \times X \times Z \times S$ be finite. Then for any $\eta > 0$, $\sigma^* : Z \times S \to \Delta(X)$ is a Proxy Equilibrium under some proxy mapping that is strongly η-close to perfect if and only if it is a **Self-Confirming Optimal** strategy.*

Moreover, if σ^* is ***strictly Self-Confirming Optimal*** *and $supp\{p(y|x,z)\} = Y$ for all $(x,z) \in X \times Z$, we can always construct the Proxy Mapping under which σ^* is a Proxy Equilibrium such that the outcome variable is perfectly measured.*

Proof. In Appendix □

The sufficiency part of the result is proven by constructing a proxy mapping that has a small probability of randomly allocating a particular realization of the true outcome vector to the proxy of an action-control combination that has zero probability under the proposed equilibrium strategy. The particular outcome vector is chosen so as to deter the DM from choosing that particular zero-probability action-control combination. The necessity part follows from a stronger version of Proposition 5, that we have convergence of beliefs to rational expectations for 'on-path' actions if the proxy mapping converges to perfect measurement.

We can apply this result to our binary policing example from Section 2.2 to analyze what strategies can arise as Proxy Equilibria for arbitrarily small measurement noise.

Example 1 (Continued)**.** *Let the payoff function of the DM be:*

$$u(y,x,s) = s(1-y) - x(1-s)$$

Assume that $\underline{s} = 0$ and $\bar{s} > 0$. Then we have $\sigma(0|\underline{s}) = 1$ in any Proxy Equilibrium. This is because at $s = 0$ it is a best response regardless of the beliefs of the DM about how x covaries with y.

There are three cases at which different strategies are Self-Confirming Optimal.

1. *If $\bar{s} \in [0, \frac{1}{2(1-\beta)})$ then we must have $\sigma(0|\bar{s}) = 1$ in any Self-Confirming Optimal strategy. This is because at $s = \bar{s}$ the following inequality must hold for $x = 1$ to be a best response.*

$$\bar{s}(2 - p_\pi(y=1|x^\bullet = 1)) - 1 \geq \bar{s}(1 - p_\pi(y=1|x^\bullet = 0)) \tag{19}$$

A strategy in which $x = 1$ is chosen with positive probability at $\bar{s}$ is one in $\sigma(0) > 0$, $\sigma(1) > 0$. Self-Confirming Optimality then requires that both actions are optimal against a system of beliefs that is correct. Since $\sigma(1) > 0$ is not if $\bar{s} \in [0, \frac{1}{2(1-\beta)})$, we have our claim.

2. *If $\bar{s} \in (\frac{1}{1-\beta}, \infty)$ then it must be the case that $\sigma(1|\bar{s}) = 1$. Since $\sigma(1|\underline{s}) = 0$ always, we must have Self-Confirming Optimality with respect to a system of beliefs for which $p_\pi(y = 1|x^\bullet = 0) = 1 - \beta$. Optimality of $x = 0$ at $s = \bar{s}$ then requires that for some beliefs $p_\pi(y = 1|x^\bullet = 1) \in [0, 1]$:*

$$\bar{s}\beta \geq \bar{s}(2 - p_\pi(y = 1|x^\bullet = 1)) - 1 \tag{20}$$

This is violated for any such beliefs if $\bar{s} > \frac{1}{1-\beta}$.

3. *If $\bar{s} \in [\frac{1}{2(1-\beta)}, \frac{1}{1-\beta}]$, then strategies in which $\sigma(1|\bar{s}) = 1$, $\sigma(1|\underline{s}) = 0$ or $\sigma(0|s) = 1$ for all s are both Self-Confirming Optimal. The first type of strategy can be implemented by the perfect measurement mapping, which is trivially arbitrarily close to perfect measurement. When $\bar{s} \in (\frac{1}{2(1-\beta)}, \frac{1}{1-\beta})$, the second type of strategy can be implemented by the following proxy mapping, which has perfect measurement of the outcome:*

$$\begin{aligned} \pi(y^\bullet = y, x^\bullet = x|y, x) &= 1 - \eta + \eta(1 - y) \\ \pi(y^\bullet = y, x^\bullet \neq x|y, x) &= \eta y \end{aligned} \tag{21}$$

This results in beliefs such that $p_\pi(y = 1|x^\bullet = 1) = 1$ and $p_\pi(y = 1|x^\bullet = 0) \to 1 - \beta$ as $\eta > 0$ tends to zero. This can then sustain the proposed strategy as a proxy equilibrium as it satisfies inequality (20) for $\bar{s} \in [0, \frac{1}{1-\beta})$. When $\bar{s} = \frac{1}{1-\beta}$ we can implement the strategy with a proxy mapping that has imperfect measurement of the outcome variable.

$$\pi(y^\bullet, x^\bullet|y, x) = (1 - \eta)1[y^\bullet = y, x^\bullet = x] + \eta\frac{1}{2}(x + (1 - x)(1 - \beta))$$

This mapping induces the exact beliefs $p_\pi(y^\bullet = 1|x^\bullet = 1) = 1$ *and* $p_\pi(y^\bullet = 1|x^\bullet = 0) = 1 - \beta$.

6 Related Literature

6.1 Relationship to Berk-Nash Equilibrium

The Berk-Nash Equilibrium of Esponda and Pouzo (2016) gives a general solution concept for games in which players have to form expectations of a mapping between actions, a signal variable and outcome variables that may depend on the actions and signals of multiple players. Each player has a set of subjective models over this mapping. Under the solution concept the expectations of the players have to be such that any subjective model that is in the support of expectation of the player minimizes the Kullback-Leibler divergence between the true distribution over outcomes and that projected by the model, weighted by that player's signal and action probabilities. This is then founded as the limit of a Bayesian learning process in which the players have a prior with their set of subjective models as the support. The true model that generates the mapping between the action, signal and outcome variables may not be in the set of subjective models and thus players may have misspecified expectations.

In general Proxy Equilibrium cannot be nested as a special case of Berk-Nash Equilibrium. The 'false' action, signal and outcome variables under Proxy Equilibrium are not compatible with the assumption that players perfectly observe the joint distribution of their signals, actions and outcomes in Berk-Nash Equilibrium. If these feedback variables are imperfectly measured, any set of subjective models cannot contain models that put probability one on the proxies being identical to the true variables. This is because the Kullback-Leibler divergence is not well defined for that model, as it would place zero probability on the event that the proxies and true variable realizations differ even though they differ with positive probability.

6.2 Relationship to Bayesian Network Equilibrium

Another related concept is the Bayesian Network Equilibrium of Spiegler (2016). In Spiegler (2016) the relationship between variables is modelled using *Directed Acyclic Graphs* (henceforth DAGs). Under the concept a DM can have a incorrect understanding of the causal relationships between the variables. The DM fits a potentially misspecified DAG to the variables in way that can distort beliefs and lead to equilibrium effects.

Bayesian Network Equilibrium can be formulated as a special case of Berk-Nash Equilibrium. Hence Proxy Equilibrium is not a special case for the same reasons as it cannot be nested as a Berk-Nash Equilibrium. Conceptually, the idea behind Proxy Equilibrium is that the simple causal structure between the variables is understood by the DM, but the DM either neglects or does not realize there are measurement problems with the variables. Bayesian Network Equilibrium instead considers cases where the DM misunderstands the causal structure, and fits the incorrect causal structure to variables that are otherwise perfectly measured.

6.3 Relationship to Self-Confirming Equilibrium

Self-Confirming Equilibrium (SCE) originates from work by Fudenberg and Levine (1993), Battigalli (1987). Under this concept players in a game only receive limited feedback on the equilibrium actions of other players and nature. In recent formulations of SCE (Battigalli et al., 2015), (Battigalli et al., 2019) there is a feedback function from true variables into a message space, with player's only observing the equilibrium distribution of messages.

Using the formulation of SCE in Esponda and Pouzo (2016), we see that the same distinction between Berk-Nash Equilibrium and Proxy Equilbrium arising from lack of knowledge of the equilibrium joint distribution of actions, signals and outcomes also holds for this version of SCE. However the most important difference is that SCE allows any belief consistent with the feedback. If we view the distribution of proxies as the feedback, Proxy Equilibrium selects a unique belief from those consistent, rather than allowing any consistent belief. For versions of SCE that allow feedback that is limited enough to be consistent with Proxy Equilibrium, this difference is enlarged as SCE then also allows an even greater set of consistent beliefs.

6.4 Other Related Literature

The strand of literature that this paper is most clearly related to is that on equilibrium solution concepts with bounded rational expectations. The work on using Bayesian Networks as a formalism to model causal misperceptions originating from Spiegler (2016) has been developed to explore interactive beliefs in games (Spiegler, 2021); political narratives (Eliaz and Spiegler, 2020), (Eliaz et al., 2022); persuasion (Eliaz et al., 2021); contract theory (Schumacher and Thysen, 2022) and deception (Spiegler, 2020). Other solution concepts in this tradition include the Cursed Equilibrium of Eyster and Rabin (2005), the Behavioural Equilibrium of Esponda (2008) and the Analogy Based Expectation Equilibrium of Jehiel (2005), Jehiel and Koessler (2008).

The Berk-Nash equilibrium of Esponda and Pouzo (2016) supplies a framework that nests many of these concepts and provides a foundation in the literature on dynamic misspecified learning. Papers in the broader misspecified learning literature have explored overconfidence about one's ability; Heidhues et al. (2018), social learning; Bohren and Hauser (2021) and connections to Berk-Nash Equilibrium; Fudenberg et al. (2021). In particular, the work of Frick et al. (2020) on fragile social learning has a similar flavour to our paper. They show that arbitrarily small misperceptions about the distribution of other player's types can generate large breakdowns in information aggregation, similar to our results on arbitrarily small imperfections in proxies leading to large distortions in beliefs.

We can see this solution concept literature as modelling players whose actions contribute to an long-run steady state distribution of the outcomes of past decisions in the same or similar situations. In contrast, there is a literature modelling players in games as extrapolating from small samples of the equilibrium behaviour of other players, the seminal work being Osborne and Rubinstein (1998) and Osborne and Rubinstein (2003). Several recent papers developing similar ideas include Salant and Cherry (2020), Patil and Salant (2020) and Gonçalves (2022).

This paper also connects to a body of work on naive inference from selected observations as a form of decision making bias. This models of sampling investors in Jehiel (2018) and elections with retrospective voters in Esponda and Pouzo (2017). Spiegler (2017) explores a procedure in which an analyst extrapolates from a dataset with partially missing information. Fudenberg et al. (2022) presents an equilibrium concept in which agents have selective recollection of their past experience. In all of these works, agents are considering a partially missing distribution. Under Proxy Equilibrium, data is not fully missing but instead distorted by measurement error.

Finally, there is a link between this paper and the literature on overconfidence in the sense of over-precision as discussed in Moore and Healy (2008). As in the case of over-precision, in Proxy Equilibrium agents underestimate the extent of the divergence of observable variables from true variables. The size of the overconfidence literature makes it impossible to cover fully here, but examples modelling over-precision specifically include applications to political ideology (Ortoleva and Snowberg, 2015), speculative bubbles in finance (Scheinkman and Xiong, 2003) and volatility in securities markets (Daniel et al., 1998).

6.5 Proxy Equilibrium vs imperfect control

We explore a variant of Proxy Equilibrium under which the DM believes they have imperfect control of the action that affects the outcome, even though they in fact have perfect control. This variant is a Bayesian Network Equilibrium and thus also a Berk-Nash Equilibrium. The differences between the equilibria possible under this variant and under Proxy Equilibrium underlines that the allowance for imperfect knowledge of the equilibrium distribution over actions discussed in Section 2.3 can have substantive effects.

Let the variable space be finite and Z be a singleton. We will focus on proxy mappings that induce perfect measurement over the outcome variable. Suppose that the DM had access to the joint distribution of the variables $(y, x, x^\bullet)$, and fits a DAG model $x \to x^\bullet \to y$ to these variables as in Bayesian Network Equilibrium. In the context of our policing example, under this variant the DM chooses a policy action x which perfectly controls police numbers. However, they only has imperfectly measured data on police numbers and believe it is possible they exert imperfect control. Given an equilibrium strategy σ, denote $\sigma(x) = \sum_{s \in S} \sigma(x|s)p(s)$. The joint density of these variables is:

$$p_\pi(y, x^\bullet, x) = p(y|x)\sigma(x)\pi(x^\bullet|y, x) \tag{22}$$

Then the conditional beliefs formed under Bayesian Network Equilibrium are:

$$\hat{q}(y|x) = \sum_{x^\bullet \in X} p_\pi(y|x^\bullet)p_\pi(x^\bullet|x) \tag{23}$$

We must have that the following holds:

$$\begin{aligned}\sum_{x \in X} \hat{q}(y|x)\sigma(x) &= \sum_{x^\bullet \in X} p_\pi(y|x^\bullet) \sum_{x \in X} p_\pi(x^\bullet|x)\sigma(x) = \sum_{x^\bullet \in X} p_\pi(y, x^\bullet) \\ &= \sum_{x^\bullet \in X} \sum_{x' \in X} \pi(x^\bullet|y, x)p(y|x')\sigma(x') = \sum_{x' \in X} p(y|x')\sigma(x')\end{aligned}$$

Thus any beliefs $\hat{q}$ arising under our DAG model must satisfy the constraint $\sum_{x \in X} \hat{q}(y|x)\sigma(x) = \sum_{x' \in X} p(y|x')\sigma(x')$. This can place a significant restriction on conditional beliefs at any x for which $\sigma(x) > 0$. If there is only one action in the support of the strategy, then this condition says that the DM must have correct conditional beliefs for that action. In contrast, Proxy Equilibrium places no such restriction.

A Appendices

A.1 Market Entry without equilibrium

The following example demonstrates that without the equilibrium requirement, against an exogenous distribution the extent of excessive entry can decrease with the bandwith parameter.

Example 2. *Let the function $m(.)$ take the following form.*

$$m(z) = \begin{cases} \frac{\frac{1}{2}k}{1-k(x-\alpha)} - \frac{1}{2}k & \text{if } z \in [0, \alpha) \\ \frac{\frac{1}{2}k+(x-\alpha)}{1-k(x-\alpha)-(x-\alpha)^2} - \frac{1}{2}k & \text{if } z \in [\alpha, 1] \end{cases} \tag{24}$$

This function is increasing in the signal z and continuous. Assume the distribution over controls $p(z)$ is uniform and the conditional distribution $p(.|z^\bullet, x = 1)$ is set exogenously to that which would be induced by the strategy $\sigma(x = 1|z) = 1$ for all z. This means that $p(.|z^\bullet, x = 1)$ is uniform in $z \in [z^\bullet - h, z^\bullet + h]$ for each $z^\bullet \in [h, 1-h]$ and uniform on $z \in [0, h)$ and $z \in (1-h, 1]$ for $z^\bullet \in [0, h)$ and $z^\bullet \in (1-h, 1]$ respectively.

The perceived utility of entry at $z \in [h, 1-h]$ is then $\int_{z-h}^{z+h} m(\tilde{z})d\tilde{z}$. This is increasing in z as $m(.)$ is increasing in z. The cut-off $\bar{z}(h)$ for which entry is a best response is given by $\int_{\bar{z}(h)-h}^{\bar{z}(h)+h} m(\tilde{z})d\tilde{z} = 0$. We can solve this to get:

$$\bar{z}(h) = \alpha - h + \frac{1}{2}k\frac{1-exp(2kh)}{exp(2kh)} + \sqrt{(h - \frac{1}{2}k\frac{1-exp(2kh)}{exp(2kh)})^2 - \frac{1-exp(2kh)}{exp(2kh)} - kh\frac{1+exp(2kh)}{exp(2kh)} - h^2}$$

We can find parameters for which there is over-entry but a lesser extent of over-entry when the bandwidth parameter is higher, in contrast to Proposition 4. Let $h_1 = 0.1$, $h_2 = 0.125$, $\alpha = 0.5$, $k = -2$. Then we have $\bar{z}(h_1) \approx 0.497 < \alpha$, $\bar{z}(h_2) \approx 0.499 < \alpha$, and $\bar{z}(h_1) < \bar{z}(h_2)$.

A.2 Proofs

Proof of Proposition 1

Proof. Denote the set of all strategies conditional on circumstance s and control z as $\Sigma(z,s)$. Define the best response correspondence, given strategy $\hat{\sigma}_\xi$ and $\xi > 0$:

$$
\begin{aligned}
& BR_\xi(\hat{\sigma}_\xi, z, s) \\
& = \{ \underset{\sigma(.|z,s)\in\Sigma(z,s)}{\arg\max} \int_X \sigma(x|z,s)[\int_Y u(y=y^\bullet, x, s) p_\pi(y^\bullet|x^\bullet = x, z^\bullet = z; \hat{\sigma}_\xi) d\mu(y^\bullet)] d\mu(x) \\
& \ \text{s.t} \ \ \sigma(x'|z,s) \geq \xi \ \ \forall x' \in X \}
\end{aligned}
$$

Stack the best response correspondences into $BR_\xi(\hat{\sigma}) = \prod_{z,s\in Z\times S} BR_\xi(\hat{\sigma}, z, s)$. Since $p_\pi(y^\bullet|x^\bullet = x, z^\bullet = z; \tilde{\sigma})$ is continuous in $\tilde{\sigma}$ and the best response correspondence is the set of maximizers over a compact set defined by a finite set of inequalities, $BR_\xi(\hat{\sigma})$ is nonempty for any $\hat{\sigma}$. Moreover due to linearity in $\sigma(x|z,s)$, $BR_\xi(.)$ convex valued and continuity of $p_\pi(y^\bullet|x^\bullet = x, z^\bullet = z; \tilde{\sigma})$ implies $BR_\xi(.)$ has closed graph. We therefore have met all the requirements of Kakutani's fixed point theorem and a fixed point exists for any $\xi > 0$, $\sigma^*_\xi \in BR_\xi(\sigma^*_\xi)$.

For any $\epsilon > 0$, we can choose $\xi > 0$ in such a way that ensures that our ξ-fixed point is an ϵ-Proxy Equilibrium. We have that $\sigma^*_\xi(x|z,s) = \xi$ for all $x \in X(z,s;\sigma^*_\epsilon)$ and (z,s). Therefore, we can choose $\xi > 0$ to ensure that $\sum_{x\in X(z,s;\sigma^*_\epsilon)} \sigma^*_\xi(x|z,s) = |X(z,s;\sigma^*_\epsilon)|\xi < \epsilon$ for all (z,s). This ensures our fixed point, which we denote σ^*_ϵ, meets the definition of ϵ-Proxy Equilibrium.

Since finiteness ensures the space of strategies Σ is compact, we can find a convergent sequence of ϵ-equilibria as $\epsilon \to 0$, $\sigma^*_\epsilon \to \sigma^*$. □

Proof of Proposition 2

Proof. We first propose a general linear solution $x(s) = \theta_0 + \theta_1 s$, which is then used to calculate the perceived expectation $\mathbb{E}[y|x^\bullet = x]$ using the properties of the normal distribution. Under the proposed best response function, the joint normal distribution of $(y, x^\bullet)$ is:

$$\begin{pmatrix} y \\ x^\bullet \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \alpha - \beta(\theta_0 + \theta_1\mu_s) \\ \theta_0 + \theta_1\mu_s \end{pmatrix}, \begin{pmatrix} \beta^2\theta_1^2\sigma_s^2 + \sigma_u^2 & -\beta\theta_1^2\sigma_s^2 \\ -\beta\theta_1^2\sigma_s^2 & \theta_1^2\sigma_s^2 + \sigma_\epsilon^2 \end{pmatrix}\right)$$

Using this we can calculate the conditional expectation of y given $x^\bullet$.

$$\mathbb{E}[y|x^\bullet] = \alpha - \beta(\theta_0 + \theta_1\mu_s) - \frac{\beta\theta_1^2\sigma_s^2}{\theta_1^2\sigma_s^2 + \sigma_\epsilon^2}(x^\bullet - \theta_0 - \theta_1\mu_s)$$

Using the utility function we then get perceived expected utility $V(x, s; \sigma) = -s\mathbb{E}[y|x^\bullet = x] - \frac{1}{2}x^2$. Solving for a maximum then gives us $x(s) = \frac{\beta\sigma_s^2\theta_1^2}{\theta_1^2\sigma_s^2+\sigma_\epsilon^2} \cdot s$. In order to have a linear equilibria, we must therefore have $\theta_0 = 0$ and $\theta_1 = \frac{\beta\sigma_s^2\theta_1^2}{\theta_1^2\sigma_s^2+\sigma_\epsilon^2}$. We can solve the latter cubic equation to get the equilibria in the statement of the proposition. ☐

Proof of Proposition 3

We first show the following fact which is used several times in our proof.

Lemma A.1. *Let $I_{[a_1,b_1]} = [a_1, b_1]$, $I_{[a_2,b_2]} = [a_2, b_2]$ be intervals in $[0, 1]$, with $a_1 > a_2$ and $b_1 > b_2$. Then for any $z_1 > z_2$ we have that:*

$$1[z_1 \in [a_1, b_1]] \cdot 1[z_2 \in [a_2, b_2]] \geq 1[z_2 \in [a_1, b_1]] \cdot 1[z_1 \in [a_2, b_2]] \tag{25}$$

Moreover, this inequality holds strictly if $z_1 \in I_{[a_1,b_1]} \setminus I_{[a_2,b_2]}$ or $z_2 \in I_{[a_2,b_2]} \setminus I_{[a_1,b_1]}$.

These facts also hold for half-open intervals $I_{[a_1,b_1)} = [a_1, b_1)$ and $I_{[a_2,b_2)} = [a_2, b_2)$.

Proof. For the right hand side of the inequality to be equal to one requires $z_2 < z_1 \leq b_2$, $z_2 \geq a_1 > a_2$, $z_1 > z_2 \geq a_1$ and $z_1 \leq b_2 < b_1$ so $z_2 \in [a_2, b_2]$ and $z_1 \in [a_1, b_1]$ and the left hand side is also equal to one.

The second part of the result holds by definition and the third part is clear by applying the arguments above again. ☐

We can then show an increasing best response property.

Lemma A.2. *Given a perceived distribution over outcomes induced by a full-support strategy σ, we have that $V(x=1,z;\sigma)$ is strictly increasing in $z \in [h, 1-h]$.*

Proof. For a given induced distribution p_π, we can use integration by parts to write the perceived utility of the DM as follows.

$$\begin{aligned} V(x=1,z;\sigma) &= \int_0^1 m(\tilde{z}) p_\pi(\tilde{z}|z^\bullet = z, x=1) d\mu(\tilde{z}) \\ &= m(1) - \int_0^1 P_\pi(\tilde{z}|z^\bullet = z, x=1) dM(\tilde{z}) \end{aligned}$$

Where $P_\pi(\tilde{z}|z^\bullet, x=1)$ is the cdf of the induced distribution and M is the Lebesgue-Stieltjes measure satisfying $M((z_l, z_h]) = m(z_h) - m(z_l)$ for any $0 \leq z_l < z_h \leq 1$. Since $m(.)$ is strictly increasing and right continuous, this measure exists. Therefore, to show the result it is enough to show $P_\pi(z|z_1^\bullet, x=1) \leq P_\pi(z|z_2^\bullet, x=1)$ for any $1-h \geq z_1^\bullet > z_2^\bullet \geq h$ and all z, with strict inequality for all z in some interval $[a,b] \subseteq [0,1]$. Our assumptions about the conditional distribution of the proxies gives us the following sequence of claims.

By Lemma A.1, we have that for any $z_1 > z_2$ and $1-h \geq z_1^\bullet > z_2^\bullet \geq h$.

$$\begin{aligned} &1[z_1 \in [z_1^\bullet - h, z_1^\bullet + h]] \cdot 1[z_2 \in [z_2^\bullet - h, z_2^\bullet + h]] \\ &\qquad \geq 1[z_2 \in [z_1^\bullet - h, z_1^\bullet + h]] \cdot 1[z_1 \in [z_2^\bullet - h, z_2^\bullet + h]] \end{aligned}$$

With strict inequality if $z_1 \in [z_1^\bullet - h, z_1^\bullet + h] \setminus [z_2^\bullet - h, z_2^\bullet + h]$ or $z_2 \in [z_2^\bullet - h, z_2^\bullet + h] \setminus [z_1^\bullet - h, z_1^\bullet + h]$, given any $1-h \geq z_1^\bullet > z_2^\bullet > h$.

Multiplying both sides by $\frac{p(z_1)\sigma(x=1|z_1)}{\int_Z p(z)\sigma(x=1|z)p(z_1^\bullet|z)d\mu(z)} \cdot \frac{p(z_2)\sigma(x=1|z_2)}{\int_Z p(z)\sigma(x=1|z)p(z_2^\bullet|z)d\mu(z)}$, we can then write:

$$p_\pi(z_1|z_1^\bullet)p_\pi(z_2|z_2^\bullet) \geq p_\pi(z_1|z_2^\bullet)p_\pi(z_2|z_1^\bullet)$$

Integrating over both sides then gives us that $P_\pi(z|z_1^\bullet, x=1) \leq P_\pi(z|z_2^\bullet, x=1)$ for any $z_1^\bullet > z_2^\bullet$ and z. By the strict inequality case above, we have that

$P_\pi(z|z_1^\bullet, x=1) < P_\pi(z|z_2^\bullet, x=1)$ for any $z \in [z_1^\bullet - h, z_1^\bullet + h] \cup [z_2^\bullet - h, z_2^\bullet + h]$. This completes the proof. $\square$

An outline of the proof is as follows. We first show why any Proxy Equilibrium with partial entry must have a cut-off structure, and give a condition that the cut-off must satisfy in terms of perceived expected utility. We then show how we can construct a sequence of ϵ-Proxy Equilibria that converge to this cut-off structure.

We consider partial entry Proxy Equilibrium in which $\sigma(x=1|z) > 0$ at a strict subset of $z \in Z^> \subset [0,1]$. For such an equilibrium to exist, we must have a sequence of ϵ^l-Proxy Equilibria, $\{\sigma^*_{\epsilon^l}\}_{l=1}^\infty$ that converge in distribution to it. Given we are considering partial entry Proxy Equilibria, for large enough l we must have that the induced belief p_π^l in the ϵ^l-Proxy Equilibrium in the sequence is such that $U(z, x=1; p_\pi^l) \leq 0$ for all $z \in [0,h]$, and that $x=1$ is a best response to p_π^l for some z. By the fact perceived utility is increasing in $z \in [h, 1-h]$, from Lemma A.2, and that a partial entry equilibrium must have no entry at some signal, there must be a cut-off $\bar{z}^{\epsilon^l} \in [h,1]$ such that a best response is $x=0$ for $z \in [0, \bar{z}^{\epsilon^l}]$ and $x=1$ for $z \in (\bar{z}^{\epsilon^l}, 1]$.

By the definition ϵ^l-Proxy Equilibrium, we must have that $\sigma^*_{\epsilon^l}(x=1|z) < \epsilon^l$ for all $z \in [0, \bar{z}^{\epsilon^l}]$ and $\sigma^*_{\epsilon^l}(x=1|z) \geq 1-\epsilon^l$ for all $z \in (\bar{z}^{\epsilon^l}, 1]$. The perceived utility of the DM at this cut-off $\bar{z}^{\epsilon^l}$ is then:

$$\int_{\bar{z}^{\epsilon^l}-h}^{\bar{z}^{\epsilon^l}} m(\tilde{z}) \frac{\sigma_\epsilon(x=1|\tilde{z})p(\tilde{z})}{\int_{\bar{z}^{\epsilon^l}-h}^{\bar{z}^{\epsilon^l}+h} \sigma_\epsilon(x=1|\hat{z})p(\hat{z})d\mu(\hat{z})} d\mu(\tilde{z}) + \\ \int_{\bar{z}^{\epsilon^l}}^{\bar{z}^{\epsilon^l}+h} m(\tilde{z}) \frac{\sigma_\epsilon(x=1|\tilde{z})p(\tilde{z})}{\int_{\bar{z}^{\epsilon^l}-h}^{\bar{z}^{\epsilon^l}+h} \sigma_\epsilon(x=1|\hat{z})p(\hat{z})d\mu(\hat{z})} d\mu(\tilde{z}) = 0$$

Thus as $l \to \infty$, if our sequence of ϵ-Proxy Equilibria converges it will converge to a Proxy Equilibrium with cut-off $\bar{z}^*$ such that $\sigma^*(x=1|z) = 0$ for $z \in [0, \bar{z}^*]$ and $\sigma^*(x=1|z) = 1$ for $z \in (\bar{z}^*, 1]$. The perceived utility at the cut-off $\bar{z}^*$ will then be:

$$\int_{\bar{z}}^{\bar{z}+h} m(\tilde{z}) \frac{p(\tilde{z})}{\int_{\bar{z}}^{\bar{z}+h} p(\hat{z}) d\mu(\hat{z})} d\mu(\tilde{z}) = 0 \tag{26}$$

We then construct strategies that can form a sequence of ϵ-Proxy Equilibria that converge to a partial entry Proxy Equilibria. These strategies have a cut-off form where $\sigma_\xi(x|z) = \xi \in (0, \frac{1}{2})$ for $z \in [0, \bar{z}]$ and $\sigma_\xi(x|z) = 1 - \xi \in (\frac{1}{2}, 1)$ for $z \in (\bar{z}, 1]$, with $\bar{z} \in [0, 1]$ as the cut-off. We can then define the following conditional density over $z \in [h, 1-h]$ given $z^\bullet = \bar{z}$.

$$g_\xi(z|z^\bullet = \bar{z}) = \frac{(1-\xi)1[\tilde{z} \in (\bar{z}, \bar{z}+h]] + \xi 1[\tilde{z} \in [\bar{z}-h, \bar{z}]]}{(1-\xi)\int_{\bar{z}}^{\bar{z}+h} p(\hat{z}) d\mu(\hat{z}) + \xi \int_{\bar{z}-h}^{\bar{z}} p(\hat{z}) d\mu(\hat{z})} p(\tilde{z}) \tag{27}$$

For any cut-off $\bar{z} \in [h, 1-h]$ and k, we can choose:

$$\xi(\bar{z}, k) = \frac{k \int_{\bar{z}}^{\bar{z}+h} p(\tilde{z}) d\mu(\tilde{z})}{k \int_{\bar{z}}^{\bar{z}+h} p(\tilde{z}) d\mu(\tilde{z}) + (1-k) \int_{\bar{z}-h}^{\bar{z}} p(\tilde{z}) d\mu(\tilde{z})} \tag{28}$$

Which is arbitrarily small for small enough $1 > k > 0$. This ensures that:

$$\int_{\bar{z}-h}^{\bar{z}} g_\xi(\tilde{z}|z^\bullet = \bar{z}) d\mu(\tilde{z}) = \frac{\xi(\bar{z}, k) \int_{\bar{z}-h}^{\bar{z}} p(\tilde{z}) d\mu(\tilde{z})}{(1 - \xi(\bar{z}, k)) \int_{\bar{z}}^{\bar{z}+h} p(\tilde{z}) d\mu(\tilde{z}) + \xi(\bar{z}, k) \int_{\bar{z}-h}^{\bar{z}} p(\tilde{z}) d\mu(\tilde{z})} = k$$

We can then write the perceived utility at $\bar{z} \in [h, 1-h]$ against the beliefs induced by strategy $\sigma_{\xi(\bar{z},k)}$ with cut-off $\bar{z} \in [h, 1-h]$ in the following way:

$$\int_0^1 m(\tilde{z}) g_{\xi(\bar{z},k)}(\tilde{z}|z^\bullet = \bar{z}) d\mu(\tilde{z}) = (1-k)\overline{U}(\bar{z}, x=1; \bar{z}) + k\underline{U}(\bar{z}, x=1; \bar{z}) \tag{29}$$

Which is a linear combination of the terms:

$$\overline{U}(\bar{z}, x=1; \bar{z}) = \int_{\bar{z}}^{\bar{z}+h} m(\tilde{z}) \frac{p(\tilde{z})}{\int_{\bar{z}}^{\bar{z}+h} p(\hat{z}) d\mu(\hat{z})} d\mu(\tilde{z}) \tag{30}$$

$$\underline{U}(\bar{z}, x=1; \bar{z}) = \int_{\bar{z}-h}^{\bar{z}} m(\tilde{z}) \frac{p(\tilde{z})}{\int_{\bar{z}-h}^{\bar{z}} p(\hat{z}) d\mu(\hat{z})} d\mu(\tilde{z}) \tag{31}$$

We show that (29) is strictly increasing in $\bar{z} \in [h, 1-h]$ by showing (30) and

(31) are strictly increasing in $\bar{z}$.

Lemma A.3. *The expressions* $\overline{U}(\bar{z}, x = 1; \bar{z})$ *and* $\underline{U}(\bar{z}, x = 1; \bar{z})$ *are strictly increasing for all* $\bar{z} \in [h, 1-h]$.

Proof. We define densities $\overline{g}(z;\bar{z}) = \frac{1_{[z\in[\bar{z},\bar{z}+h]]}p(z)}{\int_{\bar{z}}^{\bar{z}+h}p(\tilde{z})d\mu(\tilde{z})}$ and $\underline{g}(z;\bar{z}) = \frac{1_{[z\in[\bar{z}-h,\bar{z})]}p(z)}{\int_{\bar{z}-h}^{\bar{z}}p(\hat{z})d\mu(\hat{z})}$. We can then use the same steps as in the proof of Lemma A.2, applied with indicator functions $1[z \in (\hat{z}, \hat{z} + h]]$ and $1[z \in [\hat{z} - h, \hat{z}]]$ instead of $1[z \in [\hat{z} - h, \hat{z} + h]]$, to prove the result. $\square$

With these results in hand, we can then both show existence of and characterize the equilibria for this application.

Proposition 3

Proof. At any ϵ-Proxy Equilibrium, the perceived utility of the DM is increasing strictly for $z \in [h, 1-h]$ by Lemma A.2. The structure of the window form of proxy mapping means that the beliefs of the DM are identical on $z \in [0, h]$. If the DM is mixing $\sigma_\epsilon(x = 1|z) > \epsilon$ on $z \in [0, h]$, then due to increasing expected payoff on $z \in [h, 1-h]$, they must be playing $\sigma_\epsilon(x = 1|z) \geq 1 - \epsilon$ on $z \in (h, 1]$. As $\epsilon \to 0$ and $\sigma_\epsilon \to \sigma$ their perceived utility at any $z \in [0, h]$ given potential equilibrium σ is:

$$\int_0^h m(z)\frac{\sigma(x=1|z)p(z)}{\int_0^1 \sigma(x=1|\tilde{z})p(\tilde{z})d\mu(\tilde{z})}d\mu(z) + \int_h^{2h} m(z)\frac{p(z)}{\int_0^1 \sigma(x=1|\tilde{z})p(\tilde{z})d\mu(\tilde{z})}d\mu(z)$$

By the assumption that $h < \min\{\frac{\alpha}{2}, \frac{1-\alpha}{2}\}$ and m is increasing, the above expression is strictly negative. Thus for small enough ϵ at any ϵ-Proxy Equilibrium we must have $\sigma(x = 1|z) < \epsilon$ for $z \in [0, h]$.

From this argument and Lemma A.2, any ϵ-Proxy Equilibria in which $\sigma(x = 1|z) \geq \epsilon$ for some z must have some cut-off $\bar{z} \in (h, 1]$ such that $\sigma(x = 1|z) \geq \epsilon$ only if $z > \bar{z}$. The assumption $h < \min\{\frac{\alpha}{2}, \frac{1-\alpha}{2}\}$ means $\int_{1-h}^1 m(z)p(z)d\mu(z) > 0$ which ensures that any potential equilibrium strategy with cut-off $\bar{z} \geq 1 - h$ will have

$\sigma(x=1|z)=1$ as a best response for all $z\in[1-h,1]$, and thus $\sigma(x=0|z)<\epsilon$ for $z\in[1-h,1]$ in any ϵ-Proxy Equilibrium.

We construct the following cut-off ϵ-Proxy Equilibrium strategy. For any cut-off $\bar{z}\in(h,1]$, $\epsilon>0$ and $k_\epsilon\in(0,1)$, define $\xi(\bar{z},k_\epsilon)$ as in (28). Let $\sigma_\epsilon(x=1|z)=\xi(\bar{z},k_\epsilon)$ on $z\in[0,\bar{z}]$ and $\sigma_\epsilon(x=1|z)=1-\xi(\bar{z},k_\epsilon)$ on $z\in(\bar{z},1]$. We choose $k_\epsilon>0$ small enough such that $\epsilon>\sup_{\bar{z}\in[h,1-h]}\xi(\bar{z},k_\epsilon)$. Then if we can find a $\bar{z}^*$ such that a best response to the beliefs induced by σ_ϵ is $\sigma(x=1|z)=0$ on $z\in[0,\bar{z}^*]$ and $\sigma(x=1|z)=1$ on $z\in(\bar{z}^*,1]$ we have an ϵ- Proxy Equilibrium.

We have shown that the constructed strategy induces the beliefs at the cut-off $\bar{z}$ according to equation (29). We have also shown in Lemma A.3 that this expression is strictly increasing in the cut-off $\bar{z}\in[h,1-h]$. Moreover, we have that as $\epsilon\to 0$, $k_\epsilon\to 0$, so this expression converges to that in equation (26). We have (26) is strictly negative at $\bar{z}=h$ and strictly positive at $\bar{z}=1-h$. Thus we can find a small enough $\epsilon>0$ and hence $k_\epsilon>0$ such that $\int_0^1 m(\tilde{z})g_{\xi(\bar{z},k_\epsilon)}(\tilde{z}|z^\bullet=h)d\mu(\tilde{z})<0$ and $\int_0^1 m(\tilde{z})g_{\xi(\bar{z},k_\epsilon)}(\tilde{z}|z^\bullet=1-h)d\mu(\tilde{z})>0$. Since $\int_0^1 m(\tilde{s})g_{\xi(\bar{z},k_\epsilon)}(\tilde{z}|z^\bullet=\bar{z})d\mu(\tilde{z})$ is continuous and increasing in $\bar{z}\in[h,1-h]$, we can find a $\bar{z}=\bar{z}^*$ at which it is equal to zero by the intermediate value theorem. This $\bar{z}^*$ then gives us our ϵ-Proxy Equilibrium cut-off as stated above. As $\epsilon\to 0$, we can find a sequence of ϵ-Proxy Equilibria of this form that converge to that in the statement of the proposition.

Next, for contradiction consider that $\bar{z}^*\geq\alpha$. We have shown the cut-off $\bar{z}^*$ must solve the following equation.

$$\overline{U}(\bar{z}^*,x=1;\bar{z}^*)=\int_{\bar{z}^*}^{\bar{z}^*+h} m(\tilde{z})\frac{p(\tilde{z})}{\int_{\bar{z}^*}^{\bar{z}^*+h}p(\hat{z})d\mu(\hat{z})}d\mu(\tilde{z})=0$$

Then we have $\overline{U}(\bar{z}^*,x=1;\bar{z}^*)>0$ as all the probability weight in the distribution is in $z\in[\alpha,1]$, a contradiction. Thus the cut-off must be such that $\bar{z}<\alpha$.

For the final part of the proposition, we can always find a sequence of ϵ-Proxy Equilibria that converges to a Proxy Equilibrium with $x=0$ for all $z\in[0,1]$. For example, with small enough $\epsilon>0$ we can have an ϵ-Proxy Equilibrium such that the DM plays $x=1$ with probability $\epsilon>0$ on $[0,\alpha)$ and probability ϵ^2 on $[\alpha,1]$.

This induces beliefs to which $x = 0$ is a best response for all z. $\square$

Proof of Proposition 4

Proof. We are comparing the cut-off equilibrium at h_1 with the cut-off equilibrium at h_2. Consider the perceived utility at the cut-off under the equilibrium with noise h_1.

$$\overline{U}(\bar{z}(h_1), x = 1; h_1) = \int_{\bar{z}(h_1)}^{\bar{z}(h_1)+h_1} m(\tilde{z}) \frac{p(\tilde{z})}{\int_{\bar{z}}^{\bar{z}(h_1)+h_1} p(\hat{z}) d\mu(\hat{z})} d\mu(\tilde{z})$$

As this is an equilibrium cut-off, we must have that:

$$\int_{\alpha}^{\bar{z}(h_1)+h_1} m(\tilde{z})p(\tilde{z})d\mu(\tilde{z}) + \int_{\bar{z}(h_1)}^{\alpha} m(\tilde{z})p(\tilde{z})d\mu(\tilde{z}) = 0$$

If $\bar{z}(h_1)$ is fixed, then as h_1 increases to h_2, the first part of this expression that has weight on the positive part of the function $m(.)$ increases while the second part stays fixed. Thus the perceived utility at cut-off $\bar{z}(h_1)$ when the perceived distribution is induced by a strategy with cut-off $\bar{z}(h_1)$, must become positive at noise parameter $h_2 > h_1$. We have that $\overline{U}(\bar{z}(h_1), x = 1; h_2) > 0$, $\overline{U}(h_1, x = 1; h_2) < 0$ and $\overline{U}(\bar{z}, x = 1; h_2)$ is continuous in $\bar{z} \in [h_1, \bar{z}(h_1)]$. Therefore by the intermediate value theorem we can find a new cut-off $\bar{z}(h_2) < \bar{z}(h_1)$ that characterizes the positive entry equilibrium under noise parameter h_2. $\square$

Proof of Proposition 5

We prove Proposition 5 before Proposition 6 as we will use results in this section in the proof of the latter.

We first prove a sequence of lemmas. Remember that we denote $w = (y, x, z)$, the perfect measurement mapping is denoted π_δ and that $\mathcal{W}$ denotes the Borel sets of $Y \times X \times Z$. Enumerate the control, action and circumstance variables as 1, 2 and 3 respectively. Then we can denote any subset of the variable space $\{1, 2, 3\}$ by $N \subseteq 2^3$. We write the measure over the subset of true variables in N as P^N and the subset of the proxy variables in N as P^N_π. For ease of notation in this section we suppress dependence of these distributions on the strategy σ.

We can relate the distribution over all variables and the distribution over a subset N. Denote the set of Borel sets of the variable space only containing variables in N by $\mathcal{W}_N$, and $W_N \in \mathcal{W}_N$. The variable space containing the variables in any subset N is denoted as V_N.

$$P_\pi^N(W_N) = P_\pi(W_N \times \{V_{-N}\}) = \int_{Y\times X\times Z} \pi(W_N \times \{V_{-N}\}|w)p(w)d\mu(w) \tag{32}$$

Lemma A.4. *For any $\eta > 0$, if the proxy mapping is strongly η-close to perfect then it is also η-close to perfect given any distribution over the true variables P.*

In addition, for any subset of the variables N, if the proxy mapping is η-close to perfect given distribution P then we have that:

$$TV(P^N, P_\pi^N) < \eta \tag{33}$$

Proof. For the first part, we have that:

$$\begin{aligned}
TV(P, P_\pi) &= \sup_{A\in\mathcal{W}} |P(A) - P_\pi(A)| \\
&= \sup_{A\in\mathcal{W}} |\int_{Y\times X\times Z} \pi_\delta(A|w)dP(w) - \int_{Y\times X\times Z} \pi(A|w)dP(w)| \\
&= \sup_{A\in\mathcal{W}} |\int_{Y\times X\times Z} (\pi_\delta(A|w) - \pi(A|w))dP(w)| \\
&\leq \sup_{A\in\mathcal{W}} |\int_{Y\times X\times Z} \left|\pi_\delta(A|w) - \pi(A|w)\right| dP(w)| \\
&<| \int_{Y\times X\times Z} \eta dP(w)| = \eta
\end{aligned}$$

For the second part, we can show that the distance for the marginal distribution over the subset of variables N is smaller than the distance for all the variables.

$$\begin{aligned}
TV(P^N, P_\pi^N) &= \sup_{A\in\mathcal{W}_N} |P^N(A) - P_\pi^N(A)| \\
&= \sup_{A\in\mathcal{W}_N\times\{V_{-N}\}} |P(A) - P_\pi(A)| \\
&\leq \sup_{A\in\mathcal{W}} |P(A) - P_\pi(A)| = TV(P, P_\pi) < \eta
\end{aligned}$$

Where the last line follows as $\mathcal{W}_N \times \{V_{-N}\} \subset \mathcal{W}$. This completes the proof. $\square$

Lemma A.5. *Given a distribution over true variables P that admits a density p with respect to μ, let the sequence $\{\pi_n\}_{n=1}^{\infty}$ induce a sequence of distributions over the proxy variables $\{P_{\pi_n}\}_{n=1}^{\infty}$ that converges to the true distribution P in the total variation distance.*

Then there exists a subsequence $\{\pi_{n_k}\}_{k=1}^{\infty}$ that induces, for any subset of the variables N, a subsequence of densities over the proxy variables $\{p_{\pi_{n_k}}^N\}_{k=1}^{\infty}$ that converges pointwise μ almost everywhere to the true density, $p_{\pi_{n_k}}^N(w) \to p^N(w)$ for μ almost all $w \in Y \times X \times Z$.

Proof. An alternative expression for the total variation distance, given that the distributions Q_1 and Q_2 over some measure space $(\Omega, \mathcal{A})$ admit densities q_1 and q_2 with respect to some dominating measure μ, is as follows[5].

$$TV(Q_1, Q_2) = \frac{1}{2}\int_{\Omega} |q_1 - q_2| d\mu = \frac{1}{2}\|Q_1 - Q_2\|_1 \tag{34}$$

Therefore, convergence in the total variation distance is equivalent to convergence in the L1 norm. By Lemma A.4 we have that the sequence $\{\pi_n\}_{n=1}^{\infty}$ induces a sequence of distributions over the variables in N, $\{P_{\pi_n}^N\}_{n=1}^{\infty}$ which converges to the true distribution P^N in the total variation distance, and thus the L1 norm.

As each distribution in this sequence as well as the limit is assumed to admit a density function, we have that $\|p_{\pi_n}^N - p^N\|_1 \to 0$. Thus, by Theorem 13.6 (pp 465) of Charalambos and Aliprantis (2006) we have that there is a subsequence $\{p_{\pi_{n_k}}^N\}_{k=1}^{\infty}$ which converges pointwise to the true density p^N almost everywhere. $\square$

We then extend this to the induced conditional distributions over proxy variables that form the agent's beliefs under Proxy Equilibrium. For any distribution over the true variables P that admits density $p(w) = p(y|x,z)\sigma(x|z)p(z)$, let $XZ^+(\sigma) = \{(x,z) \in X \times Z : \sigma(x|z)p(z) > 0\}$.

Lemma A.6. *Given a distribution over true variables P that admits a density p with respect to μ, let the sequence $\{\pi_n\}_{n=1}^{\infty}$ induce a sequence of distributions over*

[5]See Tsybakov (2008) page 84 in Chapter 2.4 for a proof of this fact.

the proxy variables $\{P_{\pi_n}\}_{n=1}^{\infty}$ *that converges to the true distribution* P *in the total variation distance.*

Then for μ*-almost every* $(y, x, z) \in Y \times XZ^{+}(\sigma)$ *there exists a subsequence* $\{\pi_{n_k}\}_{k=1}^{\infty}$ *such that the induced subsequence of perceived conditional densities* $\{p_{\pi_{n_k}}(y^{\bullet} = y|x^{\bullet} = x, z^{\bullet} = z)\}_{k=1}^{\infty}$ *converges pointwise to the true conditional density* $p(y|x, z)$ *almost everywhere.*

Proof. By Lemma A.5 and the fact $(x, z) \in XZ^{+}(\sigma)$, for μ-almost every $(y, x, z) \in Y \times XZ^{+}(\sigma)$ both the numerator and denominator of $p_{\pi^k}(y^{\bullet} = y|x^{\bullet} = x, z^{\bullet} = z)$ converge pointwise to the true joint density as $k \to \infty$. □

This result can then be used to show the following continuity property for the perceived ex-ante expected indirect utility, under the full support assumption. Remember that under the full-support assumption $XZ^{+}(\sigma) = X \times Z$.

Proposition 5

We prove a stronger result.

Proposition 7. *For* μ *almost every* $(y, x, z) \in Y \times XZ^{+}(\sigma)$*, for any* $\epsilon > 0$ *there exists an* $\eta > 0$ *such that if the proxy mapping* π *is* η***-close to perfect*** *given a true distribution* P *that admits a density with respect to* μ *and induces a distribution over the proxy variables that satisfies the* ***full support assumption****, then* $|p_{\pi}(y^{\bullet} = y|x^{\bullet} = x, z^{\bullet} = z) - p(y|x, z)| < \epsilon$*.*

Proof. Assume for contradiction that there exists an $\epsilon > 0$ and (y, x, z) at which $p(x, z) > 0$ holds such that for any $\eta > 0$, we can find a π such that the induced distribution over proxies satisfies the full support assumption given P, $TV(P, P_{\pi}) < \eta$ and $|p_{\pi}(y^{\bullet} = y|x^{\bullet} = x, z^{\bullet} = z) - p(y|x, z)| \geq \epsilon$. Then by setting $\eta = \frac{1}{n}$, we can define a sequence $\{\pi_n\}_{n=1}^{\infty}$ that induces a sequence of distributions over the proxy variables $\{P_{\pi_n}\}_{n=1}^{\infty}$ converging to the true distribution P in the total variation distance. Then by Lemma A.6 we have that $p_{\pi_{n_k}}(y^{\bullet} = y|x^{\bullet} = x, z^{\bullet} = z) \to p(y|x, z)$, a contradiction.

Since Lemma A.6 holds for μ-almost every (y, x, z), we have that the result holds μ almost everywhere. $\square$

Since the full support assumption on P implies that $XZ^+(\sigma) = X \times Z$, this also proves Proposition 5.

Proof of Proposition 6

Proof. Remember that given a prospective strategy σ admitting a density, $XZ^+(\sigma) = \{(x, z) \in X \times Z : \sigma(x|z)p(z) > 0\}$. We first prove the following Lemma.

Lemma A.7. *For any $\epsilon > 0$, we can find an an $\eta > 0$ such that if π is strongly η-close to perfect then:*

$$\max_{(x,z,s)\in XZ^+(\sigma)\times S} |\sum_{y\in Y} u(y, x, s)p_\pi(y^\bullet = y|x^\bullet = x, z^\bullet = z; \sigma) - \sum_{y\in Y} u(y, x, s)p(y|x, z)| < \epsilon$$

Proof. Define $\overline{u} = max_{(x,z,s)\in XZ^+(\sigma)} \sum_{y\in Y} |u(y, x, s)|$. If $\overline{u} = 0$ the result holds trivially. Therefore consider $\overline{u} > 0$. By Proposition 7 and the finiteness of $Y \times X \times Z$, for any $\frac{\epsilon}{\overline{u}} > 0$ we can find an $\eta > 0$ such that if π is strongly η-close to perfect then $\max_{(y,x,z)\in Y\times XZ^+(\sigma)} |p(y|x, z) - p_\pi(y|x, z; \sigma)| < \frac{\epsilon}{\overline{u}}$. Then as required we have that for any $(x, z) \in XZ^+(\sigma)$:

$$\begin{aligned}\sum_{y\in Y} u(y, x, s)p(y|x, z) - \epsilon &< \sum_{y\in Y} u(y, x, s)p_\pi(y^\bullet = y|x^\bullet = x, z^\bullet = z; \sigma) \\ &< \sum_{y\in Y} u(y, x, s)p(y|x, z) + \epsilon\end{aligned}$$

$\square$

$\Leftarrow$ (Necessity): Clearly the second condition must hold for some beliefs q if σ^* is ever a Proxy Equilibrium. If at strategy σ^* the first condition is violated for any system of beliefs q satisfying the second condition, then either we have that for some (z, s), there are $x \in supp\{\sigma^*(.|z, s)\}$ and $x' \in supp\{\sigma^*(.|z)\}$ such that

$$\sum_{y\in Y} u(y, x, s)p(y|x, z) < \sum_{y\in Y} u(y, x', s)p(y|x', z)$$

or there exists $(z,s) \in Z \times S$, $x^{ns} \notin supp\{\sigma^*(.|z)\}$ and $x^s \in supp\{\sigma^*(.|z,s)\}$ such that for all possible conditional beliefs q

$$\sum_{y \in Y} u(y, x^s, s) p(y|x^s, z) < \sum_{y \in Y} u(y, x^{ns}, s) q(y|x^{ns}, z)$$

We show both these cases cannot hold. Define $\Delta = \sum_{y \in Y} u(y, x', s) p(y|x', z) - \sum_{y \in Y} u(y, x, s) p(y|x, z)$. By Lemma A.7, we can find $\eta > 0$ such that for any π that is strongly η-close to perfect:

$$\sum_{y \in Y} u(y, x, s) p(y|x, z) + \frac{\Delta}{4} > \sum_{y \in Y} u(y, x, s) p_\pi(y = y^\bullet | x^\bullet = x, z^\bullet = z; \sigma^*)$$
$$\sum_{y \in Y} u(y, x', s) p(y|x', z) - \frac{\Delta}{4} < \sum_{y \in Y} u(y, x, s) p_\pi(y = y^\bullet | x^\bullet = x', z^\bullet = z; \sigma^*)$$

For σ^* to be implementable as a Proxy Equilibrium at π requires that:

$$\sum_{y \in Y} u(y, x, s) p_\pi(y = y^\bullet | x^\bullet = x, z^\bullet = z; \sigma^*) \geq \sum_{y \in Y} u(y, x', s) p_\pi(y = y^\bullet | x^\bullet = x', z^\bullet = z; \sigma^*)$$

We combine this with the inequalities above to get a contradiction to the definition of Δ.

$$\frac{\Delta}{2} > \sum_{y \in Y} u(y, x', s) p(y|x', z) - \sum_{y \in Y} u(y, x, s) p(y|x, z) = \Delta > 0$$

For the second case, since $Q(x^{ns}, z)$ the space of all possible conditional beliefs given (x^{ns}, z) is compact, we have that

$$\bar{\xi} = \max_{q \in Q(x^{ns}, z)} \sum_{y \in Y} u(y, x^{ns}, s) q(y|x^{ns}, z) - \sum_{y \in Y} u(y, x^s, s) p(y|x^s, z) > 0$$

is attained. We then make the same argument as in the first step with $\bar{\xi}$ replacing Δ.

$\Rightarrow$ (Sufficiency): We first show the sufficiency of a proxy mapping where outcome variables are perfectly measured, (18) is strict and the true conditional mapping is full support. Denote the set of actions and control combinations that are not in the support of σ^* by

$$XZ^{ns}(\sigma^*) = \{(x, z) \in X \times Z : x \notin supp\{\sigma^*(.|z)\}\}$$

The set of action and control combinations in the support is then denoted $XZ^s(\sigma^*) = (X \times Z) \setminus XZ^{ns}(\sigma^*)$. Then for any (x^{ns}, z) we find $q \in Q(x^{ns}, z)$, so as to satisfy (18) strictly. For any $r \in (0, 1)$, we construct beliefs for all $(x, z) \in XZ^s(\sigma^*)$

$$\hat{q}(y|x, z) = \frac{1}{r}p(y|x, z) - \frac{1-r}{r} \sum_{(x',z') \in XZ^{ns}(\sigma^*)} q(y|x', z') \frac{1}{|XZ^{ns}(\sigma^*)|}$$

For r close to 1, since $p(y|x, z) \in (0, 1)$ for all (y, x, z) we have that $\hat{q}(y|x^{ns}, z) \in (0, 1)$ for all (x^{ns}, z). To implement these beliefs, we define the proxy mapping

$$\pi_c(y^\bullet = y, x^\bullet, z^\bullet|y, x, z) = \begin{cases} \frac{\hat{q}(y|x,z) r \sigma^*(x|z) p(z)}{\sum_{\tilde{x},\tilde{z}} p(y|\tilde{x},\tilde{z})\sigma^*(\tilde{x}|\tilde{z})p(\tilde{z})} & \text{if } (x, z) \in XZ^s(\sigma^*) \\ \frac{q(y|x,z) \frac{1-r}{|XZ^{ns}(\sigma^*)|}}{\sum_{\tilde{x},\tilde{z}} p(y|\tilde{x},\tilde{z})\sigma^*(\tilde{x}|\tilde{z})p(\tilde{z})} & \text{if } (x, z) \in XZ^{ns}(\sigma^*) \end{cases}$$

with $\pi_c(y^\bullet \neq y, x^\bullet, z^\bullet|y, x, z) = 0$. We can the see that by our definition of $\hat{q}$ that for any $(y, x, z) \in Y \times X \times Z$, $\sum_{y^\bullet} \pi_c(y^\bullet, x^\bullet, z^\bullet|y, x, z) > 0$ and:

$$\begin{aligned}
&\sum_{y^\bullet,x^\bullet,z^\bullet} \pi_c(y^\bullet, x^\bullet, z^\bullet|y, x, z) = \sum_{x^\bullet,z^\bullet} \pi_c(y^\bullet = y, x^\bullet, z^\bullet|y, x, z) \\
&= \frac{r \sum_{(x,z)\in XZ^s(\sigma^*)} \hat{q}(y|x, z)\sigma^*(x|z)p(z) + (1 - r) \sum_{(x,z)\in XZ^{ns}(\sigma^*)} q(y|x, z) \frac{1}{XZ^{ns}(\sigma^*)}}{\sum_{\tilde{x},\tilde{z}} p(y|\tilde{x}, \tilde{z})\sigma^*(\tilde{x}|\tilde{z})p(\tilde{z})} \\
&= 1
\end{aligned}$$

Thus we have that π_c is a valid proxy mapping. For any $\eta > 0$ we can then define:

$$\pi(y^\bullet, x^\bullet, z^\bullet|y, x, z) = (1 - \eta)\pi_\delta(y^\bullet, x^\bullet, z^\bullet|y, x, z) + \eta\pi_c(y^\bullet, x^\bullet, z^\bullet|y, x, z)$$

Clearly this proxy mapping is strongly-η close to perfect. This mapping induces conditional beliefs according to

$$p_\pi(y|x^\bullet, z^\bullet; \sigma^*)$$
$$= \frac{(1-\eta)p(y^\bullet|x^\bullet, z^\bullet)\sigma^*(x^\bullet|z^\bullet)p(z^\bullet) + \eta\pi_c(x^\bullet, z^\bullet|y)\sum_{x,z} p(y|x,z)\sigma^*(x|z)p(z)}{(1-\eta)\sigma^*(x^\bullet|z^\bullet)p(z^\bullet) + \eta\sum_y \pi_c(x^\bullet, z^\bullet|y)\sum_{x,z} p(y|x,z)\sigma^*(x|z)p(z)}$$

Which means that for $(x,z) \in XZ^s(\sigma^*)$, $p_\pi(y|x,z;\sigma^*) = \frac{1-\eta}{(1-\eta)+r\eta}p(y|x,z) + \frac{r\eta}{(1-\eta)+r\eta}\hat{q}(y|x,z)$ while for $(x,z) \in XZ^{ns}(\sigma^*)$ we have $p_\pi(y|x,z;\sigma^*) = q(y|x,z)$. We can choose r so that $\hat{q}(y|x,z)$ is arbitrarily close to $p(y|x,z)$. Therefore by Lemma A.7, we have that we can find r close enough to one such that σ^* is a Proxy Equilibrium due to the strict conditions (18).

Finally consider the case with only weak inequality and without full support. We can construct a proxy mapping that is strongly η-close to perfect in the same manner, with π_c instead such that for any (y,x,z) such that $x \in supp\{\sigma^*(.|z)\}$ we have perfect measurement.

$$\pi_c(y^\bullet, x^\bullet, z^\bullet|y,x,z) = \pi_\delta(y^\bullet, x^\bullet, z^\bullet|y,x,z)$$

While for any (x,z) such that $x \notin supp\{\sigma^*(.|z)\}$, chose $q \in Q(x,z)$ such that (18) holds.

$$\pi_c(y^\bullet, x^\bullet, z^\bullet|y,x,z) = q(y^\bullet|x^\bullet, x^\bullet)\hat{p}(x^\bullet, z^\bullet)$$

With $\hat{p}$ as any joint distribution such that $\hat{p}(x^\bullet, z^\bullet) > 0$ for all $(x^\bullet, z^\bullet)$. □

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